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On the mathematical structure of the equations of magnetohydrodynamic equilibrium

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Abstract. It is shown that in general the underlying mathematical structure of the system of equations which describe the static equilibrium of an ideally conducting charged fluid, in the magnetohydrodynamic approximation, depends upon the orthogonal group O(3) and the geometry of a non-flat three-dimensional Riemannian space having a constant positive curvature with the exact value of $\frac{1}{4}$.

An important mathematical device in this study has been a generalised covariant derivative which describes the pseudo-parallel displacement of a vector field in a non-flat geometry relative to a coordinate system in a flat space.

1. Introduction

The static equilibrium of an ideally conducting charged fluid, in the magnetohydrodynamic (MHD) approximation, is described by the system of equations

$$\boldsymbol{j}\times\boldsymbol{B}=\nabla P \qquad \boldsymbol{j}=\nabla\times\boldsymbol{B} \qquad \nabla\cdot\boldsymbol{B}=0 \tag{1.1}$$

where $\boldsymbol{j}$ is the current density, $\boldsymbol{B}$ is the magnetic induction and P the hydrodynamic pressure.

These equations have so far presented considerable difficulties in their analysis and few exact solutions, which describe systems of interest, have been published (e.g. Laing *et al* 1959, Woolley 1977). The work to be presented here is the first part of a study in which it will be shown that the above equations are in fact amenable to analysis in a flat three-dimensional Euclidean geometry of the kind relevant to systems of physical interest.

In general the calculations are not simple; but it is found that the equations and their solutions (at least, those obtained so far) have some interesting properties.

The first step in the analysis of the equations (1.1) is to find the conditions under which the magnetic induction $\boldsymbol{B}$ can be written in the form

$$\boldsymbol{B}=\nabla\alpha\times\nabla\beta \tag{1.2}$$

for scalar functions α and β, in a flat geometry. Some of the results of that investigation are presented in what follows.

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2. The integrability conditions

In the following work we shall use a covariant formalism in which tensor components are labelled with Greek indices whereas vectors are themselves labelled with the aid of latin indices.

We consider the equilibrium equations in the covariant form

$$\eta_{\mu\nu\sigma} j^\nu B^\sigma = P_\mu \qquad j^\mu = \eta^{\mu\nu\sigma} B_{\sigma|\nu} \qquad B^\nu{}_{|\nu} = 0 \tag{2.1}$$

where $\eta_{\mu\nu\sigma}$ is the covariant permutation tensor and covariant differentiation is with respect to the metric tensor $g_{\mu\nu}$ of a three-dimensional flat elliptic space. So $g_{\mu\nu}$ is a solution of

$$R^\mu{}_{\nu\sigma\tau} = 0 \tag{2.2}$$

where $R^\mu{}_{\nu\sigma\tau}$ is the mixed curvature tensor.

Strictly speaking, $g_{\mu\nu}$ can be any locally elliptic metric tensor; but the above conditions are necessary for systems which have physical significance.

If we represent the magnetic induction B^μ in terms of potential functions α and β according to

$$B^\mu = \eta^{\mu\nu\sigma} \alpha_\nu \beta_\sigma \tag{2.3}$$

we have the set $\{\alpha_\mu, \beta_\mu, B_\mu\}$ of linearly independent vectors in terms of which the metric tensor has the decomposition

$$g_{\mu\nu} = \frac{1}{\lambda} \{\beta^\sigma \beta_\sigma \alpha_\mu \alpha_\nu - \alpha^\sigma \beta_\sigma (\alpha_\mu \beta_\nu + \alpha_\nu \beta_\mu) + \alpha^\sigma \alpha_\sigma \beta_\mu \beta_\nu + B_\mu B_\nu\} \tag{2.4}$$

where $\lambda = B^\sigma B_\sigma$.

By now making the identification:

$$\{\alpha_\mu, \beta_\mu, B_\mu\} \equiv \{\overset{1}{a}_\mu, \overset{2}{a}_\mu, \overset{3}{a}_\mu\} \tag{2.5}$$

and representing the covariant derivatives $\overset{i}{a}_{\mu|\nu}$ in the form

$$\overset{i}{a}_{\mu|\nu} = H^i_{mn} \overset{m}{a}_\mu \overset{n}{a}_\nu \tag{2.6}$$

we can determine the equations which the coefficients H^i_{mn} must satisfy in order that the following conditions are met:

(i) B^μ, α_μ and β_μ are related by (2.3);

(ii) the equilibrium equations (2.1) are all satisfied;

(iii) covariant differentiation must commute to ensure the integrability of (2.6) in a flat geometry.

In the first instance, the most important of these conditions is (iii). Essentially it guarantees the existence of a coordinate system in terms of which the MHD equilibrium can be described. We find that this gives the system of equations

$$A^m_{n\nu|\sigma} - A^m_{n\sigma|\nu} = A^m_{s\sigma} A^s_{n\nu} - A^m_{s\nu} A^s_{n\sigma} \tag{2.7}$$

where

$$A^m_{n\mu} = H^m_{ns} \overset{s}{a}_\mu . \tag{2.8}$$

The equations (2.7) are equivalent to the condition (2.2) that the mixed curvature tensor must be identically zero in a flat space.

Now, by taking condition (i) into account, and requiring α_μ and β_μ to be gradient vectors, a lengthy calculation gives the equations (2.6) as

$$\begin{aligned} a_{\mu|\nu} &= (b_\mu - \sigma a_\mu)v_\nu + c_\mu \xi_\nu \\ b_{\mu|\nu} &= (a_\mu - \sigma b_\mu)u_\nu + c_\mu \zeta_\nu \\ c_{\mu|\nu} &= \frac{(\sigma a_\mu - b_\mu)\zeta_\nu}{1-\sigma^2} + \frac{(\sigma b_\mu - a_\mu)\xi_\nu}{1-\sigma^2}. \end{aligned} \tag{2.9}$$

In addition, the equations (2.7) are found to be equivalent to

$$\begin{aligned} (v_\mu\sqrt{1-\sigma^2})_{|\nu} - (v_\nu\sqrt{1-\sigma^2})_{|\mu} &= \frac{\xi_\mu\zeta_\nu - \xi_\nu\zeta_\mu}{\sqrt{1-\sigma^2}} \\ \xi_{\mu|\nu} - \xi_{\nu|\mu} &= (\zeta_\mu - \sigma\xi_\mu)v_\nu - (\zeta_\nu - \sigma\xi_\nu)v_\mu \\ \zeta_{\mu|\nu} - \zeta_{\nu|\mu} &= (\xi_\mu - \sigma\zeta_\mu)u_\nu - (\xi_\nu - \sigma\zeta_\nu)u_\mu \\ u_\mu + v_\mu &= \frac{\sigma_{|\mu}}{1-\sigma^2}. \end{aligned} \tag{2.10}$$

Here we have defined

$$\begin{aligned} \{a_\mu, b_\mu, c_\mu\} &\equiv \left\{\frac{\alpha_\mu}{P}, \frac{\beta_\mu}{Q}, \frac{B_\mu}{R}\right\} \\ \{\xi_\mu, \zeta_\mu, u_\mu, v_\mu\} &\equiv \left\{\frac{R}{P}A^1_{3\mu}, \frac{R}{Q}A^2_{3\mu}, \frac{P}{Q}A^2_{1\mu}, \frac{Q}{P}A^1_{2\mu}\right\} \\ \{P, Q, R\} &\equiv \{\sqrt{\alpha_\sigma\alpha^\sigma}, \sqrt{\beta_\sigma\beta^\sigma}, \sqrt{\lambda}\} \end{aligned} \tag{2.11}$$

while $\sigma = \cos\theta$, with θ the angle between α^ν and β^ν. Note that in the case where the basis triad in (2.11) is orthonormal, i.e. $\sigma = 0$, the equations (2.9) can be written quite simply in the form

$$\underset{m}{f}_{\mu|\nu} = \varepsilon_{mnr}\underset{n}{f}_\mu \underset{r}{e}_\nu \tag{2.12}$$

while (2.10) can correspondingly be written as

$$\underset{m}{e}_{\mu|\nu} - \underset{m}{e}_{\nu|\mu} = \varepsilon_{mnr}\underset{n}{e}_\mu \underset{r}{e}_\nu \tag{2.13}$$

where

$$\{\underset{1}{f}_\mu, \underset{2}{f}_\mu, \underset{3}{f}_\mu\} \equiv \{-b_\mu, a_\mu, c_\mu\} \tag{2.14}$$

$$\{\underset{1}{e}_\mu, \underset{2}{e}_\mu, \underset{3}{e}_\mu\} \equiv \{\xi_\mu, \zeta_\mu, v_\mu\} \tag{2.15}$$

$$u_\mu = -v_\mu \tag{2.16}$$

and ε_{mnr} is the cartesian permutation symbol.

The significance of the equations (2.12) and (2.13) is threefold. Firstly, the equations (2.13) are integrable and are themselves the integrability conditions for (2.12). So a solution set $\{\underset{i}{e}_\mu\}$ of vectors satisfying (2.13) allows (2.12) to be integrated and provides the set of basis vectors in (2.11).

Secondly, the integrability of (2.13) guarantees that of a set of infinitesimal transformations (see the concluding section for a brief discussion) which allow the invariant functions α and β to be determined and the equilibrium conditions (2.1) to be satisfied.

Thirdly, we find that (2.12) and (2.13) are the canonical forms of the respective equations (2.9) and (2.10) and that, so far as those equations are concerned, the function σ is subject only to the condition $|\sigma| \leq 1$. In order to prove this last assertion, we define the two sets of vectors $\{\underset{i}{M_\mu}\}$ and $\{\underset{i}{N_\mu}\}$, for $i = 1, 2, 3$, by

$$\{\underset{1}{M_\mu}, \underset{2}{M_\mu}, \underset{3}{M_\mu}\} \equiv \left\{\xi_\mu, \frac{\zeta_\mu - \sigma\xi_\mu}{\sqrt{1-\sigma^2}}, v_\mu\sqrt{1-\sigma^2}\right\} \tag{2.17}$$

$$\{\underset{1}{N_\mu}, \underset{2}{N_\mu}, \underset{3}{N_\mu}\} \equiv \left\{\frac{\xi_\mu - \sigma\zeta_\mu}{\sqrt{1-\sigma^2}}, u_\mu\sqrt{1-\sigma^2}, \zeta_\mu\right\}. \tag{2.18}$$

Then, if we add the relation

$$\underset{2}{N_\mu} + \underset{3}{M_\mu} = \frac{\sigma_{|\mu}}{\sqrt{1-\sigma^2}} \tag{2.19}$$

it is not difficult to show that the $\{\underset{i}{M_\mu}\}$ and the $\{\underset{i}{N_\mu}\}$ are both solution sets of (2.13).

Suppose that we have explicitly obtained the set $\{\underset{i}{M_\mu}\}$ as a solution of (2.13), we have

$$\begin{aligned} \xi_\mu &= \underset{1}{M_\mu} \qquad & \zeta_\mu &= \underset{1}{M_\mu}\cos\theta + \underset{2}{M_\mu}\sin\theta \\ v_\mu &= \underset{3}{M_\mu}\operatorname{cosec}\theta \qquad & u_\mu &= -\theta_{|\mu}\operatorname{cosec}\theta - v_\mu \end{aligned} \tag{2.20}$$

as a solution set of (2.10) if $\sigma \neq 0$ is defined by $\sigma = \cos\theta$ where θ can be any function of position.

The second solution set $\{\underset{i}{N_\mu}\}$ is then also determined because the relation between the solution sets is found to be

$$\underset{1}{N_\mu} = \underset{1}{M_\mu}\sin\theta - \underset{2}{M_\mu}\cos\theta \qquad \underset{1}{N_\mu} = -\underset{3}{M_\mu} - \theta_{|\mu} \qquad \underset{3}{N_\mu} = \underset{1}{M_\mu}\cos\theta + \underset{2}{M_\mu}\sin\theta \tag{2.21}$$

and this is an invariance transformation of (2.13) for any choice of θ. Now, the equations (2.9) can firstly be written in the form

$$\begin{aligned} a_{\mu|\nu} &= \left(\frac{b_\mu - \sigma a_\mu}{\sqrt{1-\sigma^2}}\right) v_\nu\sqrt{1-\sigma^2} + c_\mu\xi_\nu \\ \left(\frac{b_\mu - \sigma a_\mu}{\sqrt{1-\sigma^2}}\right)_{|\nu} &= -a_\mu v_\nu\sqrt{1-\sigma^2} + c_\mu\left(\frac{\zeta_\nu - \sigma\xi_\nu}{\sqrt{1-\sigma^2}}\right) \\ c_{\mu|\nu} &= -a_\mu\xi_\nu - \left(\frac{b_\mu - \sigma a_\mu}{\sqrt{1-\sigma^2}}\right)\left(\frac{\zeta_\nu - \sigma\xi_\nu}{\sqrt{1-\sigma^2}}\right) \end{aligned} \tag{2.22}$$

and secondly as

$$\left(\frac{a_\mu - \sigma b_\mu}{\sqrt{1-\sigma^2}}\right)_{|\nu} = -b_\mu u_\nu \sqrt{1-\sigma^2} + c_\mu \left(\frac{\xi_\nu - \sigma \zeta_\nu}{\sqrt{1-\sigma^2}}\right)$$

$$b_{\mu|\nu} = \left(\frac{a_\mu - \sigma b_\mu}{\sqrt{1-\sigma^2}}\right) u_\nu \sqrt{1-\sigma^2} + c_\mu \zeta_\nu \tag{2.23}$$

$$c_{\mu|\nu} = -b_\mu \zeta_\nu - \left(\frac{a_\mu - \sigma b_\mu}{\sqrt{1-\sigma^2}}\right)\left(\frac{\xi_\nu - \sigma \zeta_\nu}{\sqrt{1-\sigma^2}}\right).$$

The system (2.22) is equivalent to (2.12) for the solution set

$$\{\underset{1}{f_\mu}, \underset{2}{f_\mu}, \underset{3}{f_\mu}\} \equiv \left\{\frac{\sigma a_\mu - b_\mu}{\sqrt{1-\sigma^2}}, a_\mu, c_\mu\right\} \tag{2.24}$$

determined by the solution set $\{\underset{i}{M_\mu}\}$ of (2.13), as given in (2.17), while the system (2.23) is equivalent to (2.12) for the solution set

$$\{\underset{1}{\bar{f}_\mu}, \underset{2}{\bar{f}_\mu}, \underset{3}{\bar{f}_\mu}\} \equiv \left\{b_\mu, c_\mu, \frac{\sigma b_\mu - a_\mu}{\sqrt{1-\sigma^2}}\right\} \tag{2.25}$$

determined by the solution set $\{\underset{i}{N_\mu}\}$ of (2.13), as given in (2.18). Finally, it is not difficult to show that the two solution sets (2.24) and (2.25) of (2.12) are related by an invariance transformation of those equations, which is a consequence of (2.21).

Thus, either of the transformations (2.17) or (2.18) allows a solution set $\{\underset{i}{e_\mu}\}$ of the equations (2.13), in which $\sigma = 0$, to determine firstly the corresponding set $\{\xi_\mu, \zeta_\mu, u_\mu, v_\mu\}$ of vectors satisfying (2.10), with $\sigma \neq 0$ subject only to the condition $|\sigma| \leq 1$ to ensure that the angle θ between α^ν and β^ν is real. Secondly, we find that either of the solution sets in (2.17) or (2.18) provides a corresponding basis triad, through (2.12) together with either of (2.24) or (2.25), in which σ determines the angle between α^ν and β^ν as a function of position.

The relation (2.21), between (2.17) and (2.18), is a special case of a class of invariance transformations of (2.13) which will be given in the next section.

It follows that the essential properties of our equations depend upon those of (2.13); and those are the subject of the following work. In the next section we shall first obtain one of the groups of symmetry transformations which (2.13) admits and then show how those can be used to obtain solutions of the equations by means of a generation procedure which employs a class of invariance transformations.

3. The orthogonal group O(3)

We can establish the following result.

Theorem 1. When the vectors $\{\underset{i}{e_\mu}\}$, satisfying (2.13), are linearly independent they determine a set of generators for the orthogonal group O(3).

Proof. Define the set of vectors $\{\underset{i}{\lambda}^{\mu} | i = 1, 2, 3\}$ by

$$\underset{i}{\lambda}^{\mu} = \frac{1}{2k}\, \varepsilon_{imn} \eta^{\mu\nu\sigma}\, \underset{m}{e}_{\nu} \underset{n}{e}_{\sigma} \tag{3.1}$$

where

$$k = \eta^{\mu\nu\sigma} \underset{1}{e}_{\mu} \underset{2}{e}_{\nu} \underset{3}{e}_{\sigma}. \tag{3.2}$$

Then it is not difficult to obtain the following pair of results:

$$\eta_{\mu\nu\sigma} \underset{i}{\lambda}^{\nu} \underset{j}{\lambda}^{\sigma} = \frac{1}{k}\, \varepsilon_{ijn} \underset{n}{e}_{\mu} \tag{3.3}$$

$$\underset{i}{\mathscr{L}} \underset{j}{e}_{\mu} = \underset{i}{\lambda}^{\sigma} (\underset{j}{e}_{\mu|\sigma} - \underset{j}{e}_{\sigma|\mu}) + (\underset{i}{\lambda}^{\sigma} \underset{j}{e}_{\sigma})_{|\mu} = k \eta_{\mu\nu\sigma} \underset{i}{\lambda}^{\nu} \underset{j}{\lambda}^{\sigma} \tag{3.4}$$

where $\underset{i}{\mathscr{L}}$ represents Lie differentiation with respect to $\underset{i}{\lambda}^{\nu}$. With these two results we finally obtain

$$\underset{i}{\mathscr{L}} \underset{j}{\lambda}^{\nu} = \varepsilon_{ijn} \underset{n}{\lambda}^{\nu} \tag{3.5}$$

and this is the Lie algebra of the orthogonal group O(3). □

The finite transformations of the orthogonal group O(3) can be represented by matrices F_{mn} which are determined by the integrable equations

$$\underset{i}{\mathscr{L}} F_{mn} = \varepsilon_{ins} F_{ms} \tag{3.6}$$

and can be normalised to obey

$$F_{mn} F_{mr} = g_{nr} = \begin{cases} 1 & \text{if } n = r \\ 0 & \text{if } n \neq r. \end{cases} \tag{3.7}$$

By observing that $\underset{i}{e}_{\mu}$ and $\underset{i}{\lambda}^{\mu}$ satisfy

$$\underset{i}{e}_{\nu} \underset{i}{\lambda}^{\mu} = \delta^{\mu}_{\nu} \tag{3.8}$$

we obtain

$$F_{mn|\mu} = \varepsilon_{ins} \underset{i}{e}_{\mu} F_{ms}. \tag{3.9}$$

The orthogonality of the F_{mn}, together with the properties of the ε_{mnr}, then give the next result.

Theorem 2. A solution set $\{\underset{i}{e}_{\mu}\}$ of the equations (2.13) is given by

$$\underset{i}{e}_{\mu} = \tfrac{1}{2} \varepsilon_{ins} F_{mn|\mu} F_{ms}. \tag{3.10}$$

It is not difficult to show that the corresponding solution of (2.12) is given by

$$\underset{i}{f}_{\mu} = \underset{j}{\eta}_{\mu} F_{ji} \tag{3.11}$$

where the $\{\underset{j}{\eta}_{\mu}\}$ is a set of covariant-constant vectors with respect to the metric tensor $g_{\mu\nu}$.

Finally, for the purpose of conserving space, we shall state without proof the easily verified results.

Theorem 3. If $\underset{i}{\hat{e}}_{\mu} = \frac{1}{2}\varepsilon_{ins} F_{mn|\mu} F_{ms}$, and $\underset{i}{e}_{\mu}$ is any other solution of (2.13), then so is $\underset{i}{\bar{e}}_{\mu}$ given by

$$\underset{i}{\bar{e}}_{\mu} = \underset{j}{e}_{\mu} F_{ji} + \underset{i}{\hat{e}}_{\mu}. \tag{3.12}$$

Theorem 4. If $\underset{m}{f}_{\mu}$ is the solution of (2.12) determined by $\underset{m}{e}_{\mu}$, then the solution $\underset{m}{\bar{f}}_{\mu}$ corresponding to $\underset{m}{\bar{e}}_{\mu}$ is given by

$$\underset{m}{\bar{f}}_{\mu} = \underset{n}{f}_{\mu} F_{nm}. \tag{3.13}$$

As an example of theorem 3, if we take F_{mn} to be the matrix

$$F_{mn} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ -\cos\theta & 0 & \sin\theta \\ 0 & -1 & 0 \end{pmatrix}$$

then $\underset{i}{\hat{e}}_{\mu} = \frac{1}{2}\varepsilon_{iab} F_{ma|\mu} F_{mb}$ is found to be a solution of (2.13), with θ an arbitrary function, and the relation (2.21) is obtained from (3.12) if the known solution set $\{\underset{i}{e}_{\mu}\}$ is taken to be $\{\underset{i}{M}_{\mu}\}$. Further invariance transformations of the equations (2.13) which are directly related to the problem of MHD equilibrium, together with other classes of solutions, will be discussed at a later date.

For the rest of the present work we shall obtain an interesting result which follows from the proof of theorem 1; but in order to do that it is first necessary to develop the theory of a generalised covariant derivative which has turned out to be an invaluable device for obtaining results.

4. The relative covariant derivative

Let $\Gamma^{\mu}_{\nu\sigma}$ be the connection of a system of curvilinear coordinates $\{x^{\nu}\}$ in a flat space having metric tensor $g_{\mu\nu}$. Then the covariant derivative $v^{\mu}{}_{|\nu}$ of the vector field v^{μ} is given by

$$v^{\mu}{}_{|\nu} = \frac{\partial v^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\sigma} v^{\sigma}. \tag{4.1}$$

Let $a_{\mu\nu}$ be a non-singular symmetric tensor in the space of the $g_{\mu\nu}$ and let the symmetric tensor $b^{\mu\nu}$ satisfy

$$b^{\mu\sigma}a_{\sigma\nu} = \delta^{\mu}_{\nu}. \tag{4.2}$$

We then calculate the mixed tensor $\Lambda^{\mu}{}_{\nu\sigma}$ according to

$$\Lambda^{\mu}{}_{\nu\sigma} = \tfrac{1}{2}b^{\mu\tau}(a_{\nu\tau|\sigma} + a_{\sigma\tau|\nu} - a_{\nu\sigma|\tau}) \tag{4.3}$$

and define the generalised covariant derivative $v^{\mu}{}_{\cdot\nu}$ of the vector field v^{μ} by

$$v^{\mu}{}_{\cdot\nu} = v^{\mu}{}_{|\nu} + \Lambda^{\mu}{}_{\nu\sigma}v^{\sigma}. \tag{4.4}$$

It is not difficult to verify that this derivative obeys all of the axioms required of a covariant derivative. In fact, (4.4) does precisely describe covariant differentiation in the space that has the linear connection $\hat{\Gamma}^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\nu\sigma} + \Lambda^{\mu}{}_{\nu\sigma}$; but here the operation is with respect to the connection $\Gamma^{\mu}_{\nu\sigma}$ of the metric tensor $g_{\mu\nu}$. As an example, note that generally $g_{\mu\nu\cdot\sigma} \neq 0$ is given by

$$g_{\mu\nu\cdot\sigma} = -\Lambda^{\tau}{}_{\sigma\mu}g_{\tau\nu} - \Lambda^{\tau}{}_{\sigma\nu}g_{\tau\mu}. \tag{4.5}$$

So, to lower the index in (4.4) we have

$$\begin{aligned} v_{\mu\cdot\sigma} &= g_{\mu\nu}v^{\nu}{}_{\cdot\sigma} - v^{\nu}(\Lambda^{\tau}{}_{\sigma\mu}g_{\tau\nu} + \Lambda^{\tau}{}_{\sigma\nu}g_{\tau\mu}) \\ &= v_{\mu|\sigma} - \Lambda^{\tau}{}_{\sigma\mu}v_{\tau} \end{aligned} \tag{4.6}$$

which agrees with

$$\hat{\Gamma}^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\nu\sigma} + \Lambda^{\mu}{}_{\nu\sigma}. \tag{4.7}$$

A metric tensor for the space with connection $\hat{\Gamma}^{\mu}_{\nu\sigma}$ is found to be $a_{\mu\nu}$ since, by virtue of (4.3), that satisfies

$$a_{\mu\nu\cdot\sigma} = 0 \qquad b^{\mu\nu}{}_{\cdot\sigma} = 0 \tag{4.8}$$

and it is precisely this interpretation of the role of $a_{\mu\nu}$ which is of relevance for the main result of this paper.

Some particularly useful results, which are easily proved, are the following.

(I)

$$v_{\mu\cdot\nu} = \tfrac{1}{2}(v_{\mu|\nu} - v_{\nu|\mu}) + \tfrac{1}{2}\underset{u}{\mathscr{L}}a_{\mu\nu} \tag{4.9}$$

where $u^{\mu} = b^{\mu\sigma}v_{\sigma}$. So the vector u^{μ} generates an invariance transformation of the non-singular symmetric tensor $a_{\mu\nu}$ if and only if the vector $v_{\mu} = a_{\mu\sigma}u^{\sigma}$ satisfies

$$v_{\mu\cdot\nu} + v_{\nu\cdot\mu} = 0 \tag{4.10}$$

where our derivative is calculated using $a_{\mu\nu}$ as prescribed in (4.4).

(II) If ξ_{μ} is an arbitrary vector, and $a_{\mu\nu}$ is a non-singular symmetric tensor satisfying $\underset{u}{\mathscr{L}}a_{\mu\nu} = 0$, for some u^{ν}, we have with (4.9):

$$\underset{u}{\mathscr{L}}(\xi_{\mu\cdot\nu}) = \tfrac{1}{2}[(\underset{u}{\mathscr{L}}\xi_{\mu})_{|\nu} - (\underset{u}{\mathscr{L}}\xi_{\nu})_{|\mu}] + \tfrac{1}{2}\underset{u}{\mathscr{L}}\underset{\tilde{\xi}}{\mathscr{L}}a_{\mu\nu} \tag{4.11}$$

where $\tilde{\xi}^{\mu} = b^{\mu\sigma}\xi_{\sigma}$.

But

$$\underset{u}{\mathscr{L}}\underset{\tilde{\xi}}{\mathscr{L}}a_{\mu\nu} = \underset{\tilde{\xi}}{\mathscr{L}}\underset{u}{\mathscr{L}}a_{\mu\nu} + \underset{\underset{u}{\mathscr{L}}\tilde{\xi}}{\mathscr{L}}a_{\mu\nu}. \tag{4.12}$$

Thus

$$\underset{u}{\mathscr{L}}(\xi_{\mu\cdot\nu}) = \tfrac{1}{2}[(\underset{u}{\mathscr{L}}\xi_\mu)_{|\nu} - (\underset{u}{\mathscr{L}}\xi_\nu)_{|\mu}] + \tfrac{1}{2}\underset{\underset{u}{\mathscr{L}}\xi}{\mathscr{L}} a_{\mu\nu} \tag{4.13}$$

and

$$\underset{u}{\mathscr{L}}(b^{\mu\sigma}\xi_\sigma) = b^{\mu\sigma}\underset{u}{\mathscr{L}}\xi_\sigma \qquad \text{gives} \qquad \underset{u}{\mathscr{L}}(\xi_{\mu\cdot\nu}) = (\underset{u}{\mathscr{L}}\xi_\mu)_{\cdot\nu}. \tag{4.14}$$

So Lie differentiation with respect to a generator of an invariance transformation of a non-singular symmetric tensor $a_{\mu\nu}$ commutes with our derivative when that is calculated using $a_{\mu\nu}$ as prescribed in (4.4).

(III) A corollary of (II) is found to be

$$\underset{u}{\mathscr{L}}(\Gamma^\mu_{\nu\sigma} + \Lambda^\mu{}_{\nu\sigma}) = 0. \tag{4.15}$$

So $\underset{u}{\mathscr{L}}a_{\mu\nu} = 0$ requires u^ν to generate an affine motion in the space which has the connection $\hat{\Gamma}^\mu_{\nu\sigma}$.

The fundamental geometry of the space in which $a_{\mu\nu}$ can be regarded as a metric tensor is described by its curvature tensor $\hat{R}^\mu{}_{\nu\sigma\tau}$ given by

$$\hat{R}^\mu{}_{\nu\sigma\tau} = \frac{\partial}{\partial x^\sigma}\hat{\Gamma}^\mu_{\nu\tau} - \frac{\partial}{\partial x^\sigma}\hat{\Gamma}^\mu_{\nu\sigma} + \hat{\Gamma}^\rho_{\nu\tau}\hat{\Gamma}^\mu_{\rho\sigma} - \hat{\Gamma}^\rho_{\nu\sigma}\hat{\Gamma}^\mu_{\rho\tau}. \tag{4.16}$$

On substituting $\hat{\Gamma}^\mu_{\nu\sigma} = \Gamma^\mu_{\nu\sigma} + \Lambda^\mu{}_{\nu\sigma}$ into this, we obtain

$$\hat{R}^\mu{}_{\nu\sigma\tau} = R^\mu{}_{\nu\sigma\tau} + T^\mu{}_{\nu\sigma\tau} \tag{4.17}$$

where $R^\mu{}_{\nu\sigma\tau} = 0$ is the curvature tensor calculated with the flat space connection $\Gamma^\mu_{\nu\sigma}$, while

$$T^\mu{}_{\nu\sigma\tau} = \Lambda^\mu{}_{\nu\tau|\sigma} - \Lambda^\mu{}_{\nu\sigma|\tau} + \Lambda^\mu{}_{\rho\sigma}\Lambda^\rho{}_{\nu\tau} - \Lambda^\mu{}_{\rho\tau}\Lambda^\rho{}_{\nu\sigma} \tag{4.18}$$

where covariant differentiation is with respect to $g_{\mu\nu}$.

Thus, $a_{\mu\nu}$ can be regarded as a metric tensor in the space which has

$$\hat{R}^\mu{}_{\nu\sigma\tau} = T^\mu{}_{\nu\sigma\tau} \tag{4.19}$$

for its mixed curvature tensor.

Note that when $a_{\mu\nu}$ is not a solution of

$$T^\mu{}_{\nu\sigma\tau} = 0 \tag{4.20}$$

the curvature tensor $\hat{R}^\mu{}_{\nu\sigma\tau}$ will not describe a flat geometry and, in particular, if the vector field v^μ is displaced, according to $\delta v^\mu = v^\mu{}_{\cdot\sigma}\,\mathrm{d}x^\sigma = 0$, around an infinitesimal parallelogram, with sides given by $u^\nu\,\mathrm{d}s$ and $w^\nu\,\mathrm{d}t$, for some independent vectors u^ν and w^ν, it will undergo the infinitesimal change

$$\Delta v^\mu = -\hat{R}^\mu{}_{\nu\sigma\tau}v^\nu u^\sigma w^\tau\,\mathrm{d}s\,\mathrm{d}t. \tag{4.21}$$

So the differentiation described by (4.4) does not generally commute and we have

$$v_{\mu\cdot\nu\sigma} - v_{\mu\cdot\sigma\nu} = v_\tau T^\tau{}_{\mu\nu\sigma}. \tag{4.22}$$

It follows that the equations $v^\nu{}_{\cdot\sigma}u^\sigma = 0$ will in general describe a non-parallel displacement of the vector field v^μ from the point of view of the coordinate system of the $g_{\mu\nu}$. On the other hand, those equations do describe a pseudo-parallel displacement of the v^μ with reference to coordinates in the space which has the connection $\Gamma^\mu{}_{\nu\sigma} + \Lambda^\mu{}_{\nu\sigma}$.

So we have a geometrical interpretation of the derivative defined by (4.4) as describing a non-parallel displacement in the space of the metric tensor $g_{\mu\nu}$ which is equivalent to a pseudo-parallel displacement in the space in which the $a_{\mu\nu}$ can be regarded as a metric tensor. In fact, taking into account the relation (4.17) between $T^{\mu}{}_{\nu\sigma\tau}$ and the two curvature tensors, we see that the displacement in the space of the $a_{\mu\nu}$ is relative to that in the space of the $g_{\mu\nu}$. This viewpoint is justified by imagining the $g_{\mu\nu}$ to be a tensor in the space of the $a_{\mu\nu}$ and constructing the equivalent of (4.4) in that space. On using $g_{\mu\nu\cdot\sigma}$, as given in (4.5), we obtain

$$*\Lambda^{\mu}{}_{\nu\sigma} = \tfrac{1}{2} g^{\mu\tau}(g_{\nu\tau\cdot\sigma} + g_{\sigma\tau\cdot\nu} - g_{\nu\sigma\cdot\tau}) = -\Lambda^{\mu}{}_{\nu\sigma} \tag{4.23}$$

and subsequently

$$*T^{\mu}{}_{\nu\sigma\tau} = *\Lambda^{\mu}{}_{\nu\tau\cdot\sigma} - *\Lambda^{\mu}{}_{\nu\sigma\cdot\tau} + *\Lambda^{\mu}{}_{\rho\sigma} *\Lambda^{\rho}{}_{\nu\tau} - *\Lambda^{\mu}{}_{\rho\tau} *\Lambda^{\rho}{}_{\nu\sigma} \tag{4.24a}$$

gives

$$*T^{\mu}{}_{\nu\sigma\tau} = -T^{\mu}{}_{\nu\sigma\tau}. \tag{4.24b}$$

So the parallel displacement in the flat space of the $g_{\mu\nu}$ is then described relative to coordinates in the space of the $a_{\mu\nu}$. We shall therefore henceforth describe the derivative $v^{\mu}{}_{\cdot\nu}$ as 'the covariant derivative of the vector field v^{μ} with respect to the symmetric tensor $a_{\mu\nu}$ relative to the metric tensor $g_{\mu\nu}$', or 'the relative covariant derivative' for short.

5. The geometry of the integrability conditions

In the proof of theorem 1 we found the two vectors $\underset{i}{e}_{\mu}$ and $\underset{i}{\lambda}^{\mu}$ which satisfy

$$\underset{i}{e}_{\nu}\underset{i}{\lambda}^{\mu} = \delta^{\mu}{}_{\nu} \qquad \underset{i}{e}_{\nu}\underset{j}{\lambda}^{\nu} = g_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.1}$$

It follows that the two symmetric tensors

$$a_{\mu\nu} = \underset{i}{e}_{\mu}\underset{i}{e}_{\nu} \qquad b^{\mu\nu} = \underset{i}{\lambda}^{\mu}\underset{i}{\lambda}^{\nu} \tag{5.2}$$

are related by

$$b^{\mu\sigma}a_{\sigma\nu} = \delta^{\mu}_{\nu}. \tag{5.3}$$

In addition, it is seen that

$$\underset{\underset{i}{\lambda}}{\mathscr{L}} a_{\mu\nu} = 0. \tag{5.4}$$

So, according to (4.15), the orthogonal group generator $\underset{i}{\lambda}^{\nu}$ describes an affine motion in the space which has for its connection

$$\hat{\Gamma}^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\nu\sigma} + \Lambda^{\mu}{}_{\nu\sigma} \tag{5.5}$$

where $\Lambda^{\mu}{}_{\nu\sigma}$ is given by (4.3).

It is clear from (5.2) that the geometry of the space with connection $\hat{\Gamma}^{\mu}_{\nu\sigma}$ is directly a consequence of the integrability conditions (2.13). In order to describe this geometry more precisely, we first calculate the $\Lambda^{\mu}{}_{\nu\sigma}$, using (2.13), and obtain

$$\Lambda^{\mu}{}_{\nu\sigma} = \tfrac{1}{2}\underset{i}{\lambda}^{\mu}(\underset{i}{e}_{\nu|\sigma} + \underset{i}{e}_{\sigma|\nu}). \tag{5.6}$$

On substituting this into the mixed curvature tensor $\hat{R}^{\mu}{}_{\nu\sigma\tau}$, given by (4.18) and (4.19), a lengthy calculation finally gives

$$\hat{R}^{\mu}{}_{\nu\sigma\tau} = \tfrac{1}{4}(\delta^{\mu}_{\sigma} a_{\nu\tau} - \delta^{\mu}_{\tau} a_{\nu\sigma}). \tag{5.7}$$

This shows that the symmetric tensor $a_{\mu\nu}$ can be regarded as a metric tensor for a non-flat Riemannian space of three dimensions (an S_3 precisely) having a constant curvature with the exact value of $\frac{1}{4}$. Such a space can be imagined as being analogous to the surface of a familiar sphere, but having three mutually perpendicular directions at any point, instead of two, and requiring a Euclidean space of at least four dimensions for its geometrical construction—as compared with the three dimensions for the sphere of everyday experience.

This result has a number of interesting consequences for the theory of MHD equilibrium systems. In particular we have the following.

(I) The S_3 is equivalent to the configuration space of a dynamical system; it is not the flat space in which the equilibrium is supposed to exist. However, it can be transformed conformally into the flat space of the $g_{\mu\nu}$; in other words, the S_3 is conformally flat. This means that for a given $a_{\mu\nu}$ we can find a flat metric tensor $\gamma_{\mu\nu}$ together with a function φ such that

$$a_{\mu\nu} = \mathrm{e}^{2\varphi}\gamma_{\mu\nu}. \tag{5.8}$$

The function φ is determined by the equations (5.7) subject to the condition that the curvature contribution due to $\gamma_{\mu\nu}$ vanishes. A special case of this is obtained by choosing $\gamma_{\mu\nu}$ to be the coordinate metric tensor $g_{\mu\nu}$. Then the equations for φ are found to be the integrable system

$$\begin{aligned}\varphi^{\nu}{}_{|\nu} &= -\tfrac{1}{4}\mathrm{e}^{2\varphi} - H_0\,\mathrm{e}^{\varphi}\\ \varphi^{\nu}\varphi_{\nu} &= 2H_0\,\mathrm{e}^{\varphi} - \tfrac{1}{4}\mathrm{e}^{2\varphi}\\ \varphi_{\mu|\nu} &= \varphi_{\mu}\varphi_{\nu} - H_0\,\mathrm{e}^{\varphi} g_{\mu\nu}\end{aligned} \tag{5.9}$$

where H_0 is an arbitrary constant (not negative or zero).

In this case it is not difficult to show that the vector $\underset{m}{v}_{\mu}$ given by

$$\underset{m}{v}_{\mu} = \tfrac{1}{2}\mathrm{e}^{-\varphi}(2\varepsilon_{mrs}\varphi_r \underset{s}{e}_{\mu} - \underset{m}{e}_{\mu}) \qquad m = 1, 2, 3 \tag{5.10a}$$

where

$$\varphi_r = \underset{r}{\lambda}^{\sigma}\varphi_{\sigma} \tag{5.10b}$$

is a Killing vector for $g_{\mu\nu}$ and that $\underset{m}{B}^{\mu}$, given by

$$\underset{m}{B}^{\mu} = (\underset{m}{v}^{\mu}/\underset{m}{\lambda})F(\omega) + (1/\underset{m}{\lambda})\eta^{\mu\nu\sigma}\omega_{|\nu}\underset{m}{v}_{\sigma} \tag{5.11}$$

is the magnetic induction for a force-free MHD equilibrium system when $F(\omega)$ is arbitrary and the function ω is determined by the equation

$$(\omega^{|\sigma}/\underset{m}{\lambda})_{|\sigma} + 2cF/\underset{m}{\lambda}^2 + FF'/\underset{m}{\lambda} = 0. \tag{5.12}$$

Here $\underset{m}{\lambda} = \underset{m}{v}^{\sigma}\underset{m}{v}_{\sigma}$, $2c = \eta^{\mu\nu\sigma}\underset{m}{v}_{\mu}\underset{m}{v}_{\nu|\sigma}$ (not summed over m) is a constant and $F' \equiv \mathrm{d}F/\mathrm{d}\omega$. This result is a special case of a more general relation between the isometry group of the metric tensor $g_{\mu\nu}$ and MHD equilibria which is mentioned briefly in the concluding section.

(II) It can be shown that in satisfying the equilibrium equations (2.1) in the general case, the magnetic induction B^{μ} is related to the invariant functions α and β through an invariance transformation of the equations (2.10).

When the hydrodynamic pressure P only depends on one of the invariants α or β (corresponding to the physically relevant toroidal case) the transformation is precisely between solution sets of the integrability conditions (2.13) and is therefore a direct transformation of the metric tensor $a_{\mu\nu}$ of the S_3.

(III) The S_3 also admits a group of projective motions and, since any Riemannian metric is determined up to a scalar factor by its conformal and projective symmetries together, we might expect those transformations to play a role in fixing the geometry of an equilibrium through the integrability conditions (2.13) (the solutions of (2.13) and the geometry of the magnetic induction in an equilibrium system are related by a set of infinitesimal transformations which are given and discussed briefly in the concluding section).

Finally, we can mention some further results which use the relative covariant derivative with respect to $a_{\mu\nu}$ and which have applications in the solution of the integrability conditions.

(I) The linearly independent vectors satisfying (2.13) are given by the integrable equations

$$\underset{m}{e}_{\mu\cdot\nu} = \tfrac{1}{2}\varepsilon_{mrs}\underset{r}{e}_{\mu}\underset{s}{e}_{\nu} \tag{5.13}$$

(II) The equations

$$w_{\mu\cdot\nu} = -\tfrac{1}{4}wa_{\mu\nu} \tag{5.14}$$

are integrable and provide a set $\{w_i\}$ of scalar functions w_i which act as natural coordinates for the solutions of the integrability conditions (2.13).

(III) The equations

$$\zeta^{\mu}{}_{\cdot\nu} = -\tfrac{1}{4}w\delta^{\mu}_{\nu} \tag{5.15}$$

are integrable and give the set $\{\underset{i}{\zeta}^{\nu} | \underset{i}{\zeta}^{\nu} = b^{\nu\sigma}\underset{i}{w}_{\sigma}\}$ of vectors which determine the generators

$$\underset{i}{\xi}^{\mu} = \varepsilon_{imn}\underset{mn}{w\zeta}^{\mu} \tag{5.16}$$

of another group of invariance transformations of $a_{\mu\nu}$ which has the Lie algebra

$$\underset{\underset{i}{\xi}}{\mathcal{L}}\underset{j}{\xi}^{\mu} = \varepsilon_{imj}c_{mp}\underset{p}{\xi}^{\mu} \tag{5.17}$$

where the c_{mp} is a symmetric matrix of arbitrary constants.

A relevant point here, which remains to be studied, is that the maximum order of the group of motions of an S_3 is exactly 6. So it might well be that (3.5) and (5.17) give the Lie algebras of subgroups of the same group.

In any event, the existence of symmetry groups of $a_{\mu\nu}$ provides a starting point for the study of the solution classes of the integrability conditions (2.13) and, possibly, a classification of MHD equilibrium systems.

The results presented so far have all been obtained on the basis of the assumption that the solution vectors $\underset{m}{e}_\mu$ of (2.13) are linearly independent and, for the sake of completeness, it is necessary to include the special case in which the $\underset{m}{e}_\mu$ are linearly dependent.

6. The case of linear dependence

When the $\underset{m}{e}_\mu$, satisfying (2.13), are linearly dependent it is found that they must have the general form

$$\underset{i}{e}_\mu = \underset{i}{e}_{X^1} X^1_\mu + \underset{i}{e}_{X^2} X^2_\mu \tag{6.1}$$

where the X^1 and X^2 are arbitrary independent functions of the x^ν, X^1_μ and X^2_μ are gradient vectors and the $\underset{i}{e}_{X^1}$ and $\underset{i}{e}_{X^2}$ are functions of X^1 and X^2 only which are determined by (2.13) in the form

$$\frac{\partial}{\partial X^n} \underset{i}{e}_{X^m} - \frac{\partial}{\partial X^m} \underset{i}{e}_{X^n} = \varepsilon_{irs} \underset{r}{e}_{X^m} \underset{s}{e}_{X^n}. \tag{6.2}$$

The symmetric tensor

$$a_{\mu\nu} = \underset{i}{e}_\mu \underset{i}{e}_\nu \tag{6.3}$$

in this case determines the metric form for a Riemannian space of at most two dimensions in which X^1 and X^2 act as local coordinates. If we define the vector $\underset{i}{e}_\mu$ in that space by

$$\underset{i}{e}_\mu = (\underset{i}{e}_{X^1}, \underset{i}{e}_{X^2}) \equiv (\underset{i}{e}_1, \underset{i}{e}_2) \tag{6.4}$$

it is possible to obtain the following results.

(I) The metric tensor is given by

$$a_{\mu\nu} = \underset{i}{e}_\mu \underset{i}{e}_\nu \tag{6.5}$$

where Greek indices now take values in $\{1, 2\}$.

(II) The curvature tensor is given by

$$R_{\mu\nu\sigma\tau} = \kappa \eta_{\mu\nu} \eta_{\sigma\tau} \tag{6.6a}$$

where

$$\kappa = \tfrac{1}{2} \varepsilon_{mnr} \eta^{\sigma\tau} u_m u_{n|\sigma} u_{r|\tau} \tag{6.6b}$$

$$u_n = \eta^{\sigma\tau} \underset{n}{e}_{\sigma|\tau} \tag{6.6c}$$

while $\eta^{\mu\nu}$ is the contravariant permutation tensor and covariant differentiation is with respect to $a_{\mu\nu}$.

(III) The u_n satisfy the integrable equations

$$u_n u_n = 1 \qquad \underset{w_m}{\mathscr{L}} u_n = \kappa \varepsilon_{nmr} u_r \qquad w^\mu_m = \eta^{\mu\nu} u_{m|\nu}. \tag{6.7}$$

It is found that these geometries in particular allow the MHD equilibrium conditions to be solved and such solutions will be discussed at a later date.

It is worth mentioning that the linearly dependent solutions of (2.13) appear to be related to an invariant of the magnetic induction which does not occur so readily when the $\underset{m}{e}_\mu$ are independent. The precise distinction between the two classes of solution of (2.13)—in respect of resulting MHD equilibria—remains to be studied.

7. Conclusions

The essence of the work which has been presented here is that if the magnetic induction B^μ in an equilibrium system is to be represented in the form

$$B^\mu = \eta^{\mu\nu\sigma}\alpha_\nu\beta_\sigma$$

where α_ν and β_ν are gradient vectors, then the basis of the mathematics must depend upon the integrability conditions (2.13)—if the equilibrium is to exist in a physically relevant geometry—and those in turn impose a geometrical structure on the system which in general depends upon the orthogonal group O(3) and an S_3 having the precise value of $\frac{1}{4}$ for its curvature.

Of course, it is necessary that B^μ is representable in the above form firstly because we must have $B^\nu{}_{|\nu} = 0$ and, secondly, because the hydrodynamic pressure P, in an equilibrium, will be a function of α and β due to the field lines of B^μ forming the invariant surfaces on which P takes constant values.

There is, in addition, another more subtle way in which the integrability conditions (2.13) impose a structure on the equilibrium itself which, due to lack of space we have not gone into in detail here. Briefly, it can be shown that the equilibrium field vectors, which satisfy (1.1), determine the set of infinitesimal transformations

$$\underset{\tilde{u}}{\mathscr{L}}\tilde{v}^\nu = \frac{\omega}{\lambda}\tilde{B}^\nu \qquad \underset{\tilde{B}}{\mathscr{L}}\tilde{v}^\nu = -\frac{P_\alpha}{\lambda}\tilde{B}^\nu \qquad \underset{\tilde{B}}{\mathscr{L}}\tilde{u}^\nu = -\frac{P_\beta}{\lambda}\tilde{B}^\nu$$

where

$$\tilde{B}^\nu = \frac{B^\nu}{\lambda} \qquad \tilde{u}^\nu = \frac{1}{\lambda}\eta^{\nu\mu\sigma}B_\mu\alpha_\sigma \qquad \tilde{v}^\nu = \frac{1}{\lambda}\eta^{\nu\mu\sigma}\beta_\mu B_\sigma$$

$$P_\alpha = \frac{\partial P}{\partial \alpha} \qquad P_\beta = \frac{\partial P}{\partial \beta} \qquad \omega = j^\sigma B_\sigma .$$

These equations can also be derived from the system (2.9) or (2.12), when α_ν and β_ν are gradient vectors and the pressure balance in (2.1) is true. In fact, they are the link between the geometry of the equilibrium field vectors and that of the S_3 because it is found that the quantities P_α/λ, P_β/λ and ω/λ must themselves be determined by the $\underset{m}{e}_\mu$ in order that the Jacobi identity

$$\underset{\tilde{u}}{\mathscr{L}}\underset{\tilde{v}}{\mathscr{L}}\tilde{B}^\nu + \underset{\tilde{v}}{\mathscr{L}}\underset{\tilde{B}}{\mathscr{L}}\tilde{u}^\nu + \underset{\tilde{B}}{\mathscr{L}}\underset{\tilde{u}}{\mathscr{L}}\tilde{v}^\nu = 0$$

is true for the infinitesimal transformations to be integrable. So the equilibrium system is determined along with the conditions which dictate its geometrical structure in a flat space.

This intimate link between an MHD equilibrium system and the geometry of the space in which it exists is seen in a simple way when it is realised that a Killing vector, satisfying Killing's equation

$$\underset{v}{\mathcal{L}} g_{\mu\nu} = v_{\mu|\nu} + v_{\nu|\mu} = 0$$

is a possible magnetic induction for an MHD equilibrium system. In fact, it can be shown that the solutions of (2.1) which satisfy

$$\underset{v}{\mathcal{L}} B^{\mu} = 0$$

where v^{ν} is a Killing vector, include the cylindrical, axisymmetric toroidal and helical systems which have been found by ansatz and studied in detail already (e.g. Laing *et al* 1959, Woolley 1975).

Our final conclusion might well be of general interest. The basic results presented here really apply to a much wider class of systems than MHD equilibria. For example, the vector B^{μ} in (2.3) can in principle represent any divergence-free vector field for which that description is applicable. Thus, B^{μ} could describe an equilibrium current density—or any other vector field; it is only when the equilibrium conditions (2.1) have been satisfied that the field is particularised.

If we define an almost-arbitrary divergence-free vector field as one for which the vectors in (2.13) are linearly independent, our main result amounts to a proof of the following theorem.

Theorem. The geometry of an almost-arbitrary divergence-free vector field in a three-dimensional flat elliptic space is determined by the orthogonal group O(3) and the geometry of a three-dimensional Riemannian space having a constant curvature with the exact value of $\frac{1}{4}$.

References

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